Disturbances in a Multi-Velocity Plasma*

J. R. PIERCE† AND J. A. MORRISON†

Summary—The single-velocity treatment of ac disturbances in electron flow is often accurate enough even when the flow actually has an infinitely broad but peaked velocity distribution, such as a Maxwellian or a bell-shaped distribution. In seeking a better approximation, one is tempted to expand in terms of the moments of the velocity distribution and to disregard terms beyond that involving the second moment. This leads to a dispersion equation describing waves which strictly do not exist. A linearized analysis by transform methods predicts a field which oscillates in an exponentially damped manner. This is not the oscillation of a normal mode, however, for various velocity classes of the charge distribution have ac densities which grow with time. This indicates that the linear expression will hold over a finite interval only.

A simple example which compares a multi-stream analysis with an analysis based on the charge density in phase space (the distribution function) indicates the same sort of failure of both of the linearized theories at large times. Thus, it appears that the failure is characteristic of the problem rather than of the method of solution. The method of solution is a matter of choice.

INTRODUCTION

WORKERS in the field of electron tubes have had a number of years to deal with problems concerning the multivelocity nature of actual electron flow. Some of these problems are concerned with noise in electron beams. Here the crucial effect of multivelocity flow lies in the fact that the spread in electron velocities, together with the discrete nature of electrons, leads to two independent sources of noise near the cathode, and these noise sources give rise to noise space-charge waves in the electron beam.[1,2]

In noise calculations, one can sometimes get sufficiently accurate results by disregarding the effect of the multivelocity nature of the electron flow in dealing with the propagation of noise disturbances along the electron beam, but this can lead to serious errors in the low-velocity region near the cathode.[3] Thus, we are faced with the questions: How can we best take into account the actual velocity distribution? Is there any simple approximation which will help us? It is the purpose of this paper to review our present knowledge of this matter. In the course of this review, we shall exhibit the general features of the solution of multi-velocity problems by both the multi-stream approach and the distribution-function or density-function approach.[4]

We shall consider the case of disturbances in an infinite cloud of electrons; we may imagine their average charge to be neutralized by a fixed positive charge. We shall

---

* Manuscript received by the PGED, June, 1957; revised manuscript received, December 9, 1958.

† Bell Telephone Labs., Inc., Murray Hill, N. J.

[1] J. R. Pierce, "A theorem concerning noise in electron streams," *J. Appl. Phys.*, vol. 25, pp. 931–933; August, 1954.

[2] H. A. Haus and F. N. H. Robinson, "The minimum noise figure of microwave beam amplifiers," PROC. IRE, vol. 43, pp. 981–991; August, 1955.

[3] A. E. Siegman, D. A. Watkins and Hsung-Cheng Hsieh, "Density function calculations of noise propagation on an accelerated multi-velocity beam," *J. Appl. Phys.*, vol. 28, pp. 1138–1148; October, 1957.

[4] A. E. Siegman, "Analysis of multi-velocity electron beams by the density-function method," *J. Appl. Phys.*, vol. 28, pp. 1132–1138; October, 1957.

consider disturbances in which the fields vary in the z direction only. The treatment will be Newtonian and electrostatic.

Waves in Multi-Velocity Flow?

Let us consider an undisturbed cloud of electrons in which the average charge density associated with electrons whose z-component of velocity lies in the range du at u is $-d\rho_0$. Let $-\rho_0$ be the total charge density for electrons of all velocities. Then we can write

$$d\rho_0 = \rho_0 P(u)\, du. \tag{1}$$

Here $P(u)$ describes the velocity distribution of the electron charge density. It is clear that the form of the expression has been chosen so that

$$\int_{-\infty}^{\infty} P(u)\, du = 1. \tag{2}$$

How shall we take a disturbance in such an electron cloud into account? Let us consider a case in which all ac quantities vary with time and position as

$$e^{-j\beta z} e^{j\omega t}.$$

It is easily shown that for electrons which have a velocity *u in the absence of the disturbance*, the ac convection current di associated with the field E is given by

$$di = \frac{-j\omega(e/m)E\, d\rho_0}{(\omega - \beta u)^2}. \tag{3}$$

To get the total convection current i we must integrate over all velocities; thus

$$i = -j\omega(e/m)\rho_0 E \int_{-\infty}^{\infty} \frac{P(u)}{(\omega - \beta u)^2}\, du. \tag{4}$$

A plausible approximation has been proposed.[5] Let us assume that βu is very small compared with ω, and expand the denominator in the integral so that

$$i = \frac{-j(e/m)\rho_0 E}{\omega} \cdot \int_{-\infty}^{\infty} (1 + 2(\beta u/\omega) + 3(\beta u/\omega)^2 + \cdots)P(u)\, du$$

$$i = \frac{-j(e/m)\rho_0 E}{\omega} (1 + 2(\beta/\omega)\langle u\rangle + 3(\beta/\omega)^2\langle u^2\rangle + \cdots). \tag{5}$$

This procedure is strictly justified if $|\beta u/\omega|$ is less than unity for the greatest value of $|u|$ in the velocity distribution. However, it is dubious to use this expansion in connection with infinite velocity distributions.

If we assume $\langle u\rangle = 0$ (zero average velocity) and disregard all terms beyond the term in $\langle u^2\rangle$, as we might if $|\beta u/\omega|$ were very small for the largest $|u|$ in the distribution, we obtain

$$i = \frac{-j(e/m)\rho_0 E}{\omega} (1 + 3(\beta/\omega)^2\langle u^2\rangle). \tag{6}$$

Now let us assume that the total electric field is due to the convection current i. As the total current density must be zero

$$i + j\omega\epsilon E = 0. \tag{7}$$

From (6) and (7) we obtain

$$\omega^2 = \omega_p^2(1 + 3(\beta/\omega)^2\langle u^2\rangle). \tag{8}$$

The quantity ω_p, called the plasma frequency, is

$$\omega_p^2 = \frac{(e/m)}{\epsilon}\rho_0. \tag{9}$$

Eq. (8) can be rewritten

$$\beta = \pm\left(\frac{\omega^2}{3\langle u^2\rangle}\right)^{1/2}\left(\left(\frac{\omega}{\omega_p}\right)^2 - 1\right)^{1/2}. \tag{10}$$

This relation is usually known as the dispersion relation of Bohm and Gross.[5]

It appears that a disturbance of a particular phase constant (and wavelength) is associated with an excitation of any frequency ω greater than the plasma frequency. If the mean square velocity $\langle u^2\rangle$ approaches zero, however, the range of frequencies corresponding to any finite range of phase constants β approaches zero at $\omega = \omega_p$.

To explore this matter, let us assume a particular velocity distribution which is simple enough to handle analytically. Let

$$d\rho_0 = \frac{\rho_0 u_0\, du}{\pi(u^2 + u_0^2)}. \tag{11}$$

Here $-\rho_0$ is the average charge density for all velocity classes of electrons. Thus, from (3) and (11), the total convection current will be

$$i = -j(e/m)(\rho_0 u_0/\pi)(\omega/\beta^2)E \cdot \int_{-\infty}^{\infty} \frac{du}{(u + ju_0)(u - ju_0)(u - \omega/\beta)^2}. \tag{12}$$

The integral can be evaluated by contour integration. As the integrand goes to zero at infinity in all directions in the complex u plane, we can close the contour either above or below the real axis. If we assume ω to have a positive imaginary part, it is most convenient to close the contour below the real axis, including only the pole at

$$u = -ju_0. \tag{13}$$

This leads to

$$i = \frac{-j(e/m)\rho_0\omega E}{(\omega + j\beta u_0)^2}. \tag{14}$$

If, however, we assume ω to have a negative imaginary part, we obtain

$$i = \frac{-j(e/m)\rho_0\omega E}{(\omega - j\beta u_0)^2}. \tag{15}$$

At this point, we look for a solution of both (7) and (14) or (15). From (7) and (14) we obtain

---

[5] D. Bohm and E. P. Gross, "Theory of plasma oscillations: A. Origin of medium-like behavior, *Phys. Rev.*, vol. 75, pp. 1851–1876; June 15, 1949.

$$\omega = \pm\omega_p - j\beta u_0. \tag{16}$$

The imaginary part of ω as given by (16) is negative, contradicting the assumption which led to (14), that the imaginary part of ω was positive. We arrive at a like contradiction if we use (15) and (7). Thus, there are no normal modes in which all the ac quantities vary as exp $(j\omega t - j\beta z)$.

We might think that we fail to find waves of the sort described by (10) because for the velocity distribution (11), $\langle u^2 \rangle$ is infinite. If we believed relation (8), this would lead to an infinite frequency of oscillation for an oscillation of finite wavelength ($\beta \neq 0$). However, we shall show that despite the infinite mean square velocity, the distribution exhibits a calculable and reasonable behavior following excitation. Further, this behavior is similar to that which Landau[6] has found for a plasma having a Maxwellian velocity distribution. Also, Walker[7] has studied the general case of waves in streams with an arbitrary velocity distribution. He finds that when the velocity distribution has a finite width, there are unattenuated waves having phase velocities lying outside the range of electron velocities. When the velocity distribution has an infinite width, there are no inattenuated waves; there may in this case be growing and decaying waves if the velocity distribution has two or more peaks with sufficiently deep valleys between them.

Thus the dispersion relation (10), although plausible, has no strict significance in the case of infinite velocity distributions, although it does give the correct phase constant for small values of $(\beta/\omega)^2\langle u^2 \rangle$ in the case of the Maxwellian velocity distribution.[6] We may ask, however, what sort of disturbances can exist in a plasma with an infinite, single-peaked velocity distribution?

We are unable to treat this problem in its true nonlinear form. It is possible to obtain a linearized solution, presumably valid for small disturbances. We will find, however, that linearized solutions of electron-flow problems can be in error at large enough times even when the initial amplitude of excitation is small.

A PLASMA OF INTERACTING PARTICLES

Let us consider what happens when a plasma of particles with the velocity distribution of (11) is acted on by an impulsive force with a sinusoidal spatial distribution and by space-charge forces. For convenience, we will write the initial force on a particle as $-eF$, where we may think of F as a nonelectrical quantity which is introduced merely to impart an initial velocity to the electrons. We will assume that F has the form

$$F = \frac{-v_p}{(e/m)} e^{-j\beta z}\delta(t), \tag{17}$$

where $\delta(t)$ is a delta function. This force will give a particle at $z = 0$ an additional velocity v_p at $t = 0$. Considering the stream corresponding to an initial velocity u, we have, according to the linearized theory,

$$\begin{aligned} \frac{\partial}{\partial t}(d\rho) &= -\frac{\partial(di)}{\partial z}, \\ (di) &= (d\rho)u - (d\rho_0)v, \\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial z} &= -\frac{e}{m}(E + F), \\ i + \epsilon\frac{\partial E}{\partial t} &= 0, \end{aligned} \tag{18}$$

where $(d\rho)$, (di) and v are the perturbation density, convection current, and particle velocity, respectively. The last equation of the group asserts that the total current density is zero. If we let $d\rho = d\rho^* e^{-j\beta z}$, etc., (18) becomes, upon taking Laplace transforms,

$$\begin{aligned} (S - j\beta u)\,\overline{v^*} &= v_p - \frac{e}{m}\overline{E^*}, \\ \overline{i^*} + S\epsilon\,\overline{E^*} &= 0, \\ S\overline{(d\rho^*)} &= j\beta\overline{(di^*)}, \\ \overline{(di^*)} &= \overline{(d\rho^*)}u - (d\rho_0)\overline{v^*}, \end{aligned} \tag{19}$$

where S is the transform parameter. Eliminating $\overline{v^*}$, $\overline{(d\rho^*)}$ and $\overline{E^*}$, we obtain

$$\overline{(di^*)} = -(d\rho_0)\frac{\left[Sv_p + \frac{e}{m\epsilon}\overline{i^*}\right]}{(S - j\beta u)^2}. \tag{20}$$

Inverting (20), and using the convolution theorem,[8]

$$di^* = -(d\rho_0)\Bigg(v_p(1 + j\beta ut)e^{j\beta ut} + \frac{e}{m\epsilon}\int_0^t i^*(\tau)(t - \tau)e^{j\beta u(t-\tau)}\,d\tau\Bigg). \tag{21}$$

Integrating (21) with respect to u from $-\infty$ to $+\infty$, where $d\rho_0$ is given by (11),

$$i^* = -\rho_0\Bigg(v_p(1 - \beta u_0 t)e^{-\beta u_0 t} + \frac{e}{m\epsilon}\int_0^t i^*(\tau)(t - \tau)e^{-\beta u_0(t-\tau)}\,d\tau\Bigg). \tag{22}$$

This is an integral equation for i^* which may be solved by taking Laplace transforms,

$$\overline{i^*} = \frac{-\rho_0\left[Sv_p + \frac{e}{m\epsilon}\overline{i^*}\right]}{(S + \beta u_0)^2}\,;\ \overline{i^*} = \frac{-\rho_0 S v_p}{[(S + \beta u_0)^2 + \omega_p^2]}, \tag{23}$$

---

[6] L. Landau, "On the vibrations of the electronic plasma," *J. Phys. USSR*, vol. 10, no. 1, pp. 25–34; 1946.

[7] L. R. Walker, "The dispersion formula for plasma waves," *J. Appl. Phys.*, vol. 25, pp. 131–132; January, 1954.

[8] W. Magnus and F. Oberhettinger, "Formulas and Theorems for the Functions of Mathematical Physics," Chelsea Publishing Co., New York, N. Y., ch. 8, sec. 2; 1954.

where ω_p is the plasma frequency as defined in (9). From the second equation of (19), it then follows that

$$\overline{E^*} = \frac{\rho_0 v_p}{\epsilon} \cdot \frac{1}{[(S + \beta u_0)^2 + \omega_p^2]}. \tag{24}$$

Inverting (24) we find

$$E^* = \frac{\rho_0 v_p}{\omega_p \epsilon} \sin(\omega_p t) e^{-\beta u_0 t}. \tag{25}$$

This bears not the slightest resemblance to the result of Bohm and Gross[5] (8). Instead of an oscillation of frequency other than ω_p, we have a damped oscillation of frequency ω_p.

Clearly, although the field has only this exponentially decaying sinusoidally oscillating component, currents carried by streams of various velocities must have a more complicated behavior.

Let us then consider the charge density $d\rho$ associated with the electrons of charge density $d\rho_0$ and of average velocity u in the range du. There will be two components of the charge density $d\rho$ associated with these electrons, that due to the impulsive force F, and that due to the subsequent electric field E, which is zero for $t < 0$. From (19) and (20),

$$\overline{(d\rho^*)} = -j\beta\, d\rho_0 \frac{\left[v_p - \frac{e}{m}\overline{E^*}\right]}{(S - j\beta u)^2} = \overline{(d\rho_1^*)} + \overline{(d\rho_2^*)}. \tag{26}$$

The first component of $\overline{(d\rho^*)}$ is that caused by the impulsive force F at $t = 0$,

$$\overline{(d\rho_1^*)} = \frac{-j\beta v_p (d\rho_0)}{(S - j\beta u)^2}. \tag{27}$$

Inverting (27), we obtain

$$d\rho_1 = -j\beta v_p t (d\rho_0) e^{j\beta u t} e^{-j\beta z}. \tag{28}$$

From equations (26) and (24),

$$\begin{aligned}\overline{(d\rho_2^*)} &= \frac{j\beta(d\rho_0)\frac{e}{m}\overline{E^*}}{(S - j\beta u)^2} \\ &= \frac{j\beta v_p (d\rho_0)\omega_p^2}{(S - j\beta u)^2[(S + \beta u_0)^2 + \omega_p^2]}.\end{aligned} \tag{29}$$

Expressing the right-hand side of (29) in partial fractions and inverting, we find the part of the charge density due to the field E is

$$\begin{aligned}d\rho_2 = {} & +jv_p\, d\rho_0 \beta t e^{j\beta u t}\left[\frac{\omega_p^2}{\omega_p^2 - [\beta(u - ju_0)]^2}\right] e^{-j\beta z} \\ & - \frac{1}{2} v_p\, d\rho_0 \beta \omega_p e^{-\beta u_0 t}\left[\frac{e^{j\omega_p t}}{[\omega_p - \beta(u - ju_0)]^2}\right. \\ & \left. - \frac{e^{-j\omega_p t}}{[\omega_p + \beta(u - ju_0)]^2}\right] e^{-j\beta z} \\ & + 2v_p\, d\rho_0 \beta^2 \omega_p^2 e^{j\beta u t} \frac{(u - ju_0)e^{-j\beta z}}{([\beta(u - ju_0)]^2 - \omega_p^2)^2}.\end{aligned} \tag{30}$$

From (28) and (30) we see that there is a component of charge density $d\rho_a$ associated with electrons in the velocity range du at u which increases linearly with time; it is

$$d\rho_a = j\, d\rho_0 e^{-j\beta z} \beta v_p t e^{j\beta u t}\left[\frac{[\beta(u - ju_0)]^2}{\omega_p^2 - [\beta(u - ju_0)]^2}\right]. \tag{31}$$

We note that (31) says that when

$$\frac{\beta v_p t \,|\, \beta(u + ju_0) \,|^2}{|\, \omega_p^2 - [\beta(u - ju_0)]^2 \,|} = 1 \tag{32}$$

the peak value of the ac charge density $d\rho_a$ will be equal to the dc charge density $d\rho_0$ of the electrons of the stream with which it is associated. Clearly, the approximation of linearity will have failed before this time.

Let us consider the case in which the wavelength of the disturbance is very short, so that

$$|\, \beta(u + ju_0) \,| \gg \omega_p. \tag{33}$$

If this is so, (25) will hold only when

$$\beta v_p t \ll 1. \tag{34}$$

Thus, for this case, (25) will hold only until the oscillation has decayed to a point such that

$$e^{-\beta u_0 t} \gg e^{-u_0/v_p}. \tag{35}$$

The smaller the initial velocity of excitation v_p compared with the breadth of the velocity distribution, the further the oscillation decays in amplitude before its behavior is seriously affected by nonlinearities.

The number of cycles of oscillation which take place before nonlinearities become important is also of interest. When β is large and (33) holds, this number N is

$$N = \frac{\omega_p t}{2\pi} \ll \frac{\omega_p}{2\pi\beta v_p}. \tag{36}$$

As a practical matter, nonlinearities may not be particularly important. For a plasma with a velocity spread comparable to the Maxwellian spread for common temperatures, the rate of exponential decay will be small, and for times of interest the behavior will be close to that of a single-velocity plasma. Further, there is every reason to believe that in the exact solution, as in the linearized solution, the field will become very small at a sufficiently great time subsequent to excitation.

It is, however, interesting to note that the exponentially decaying solution of the linearized problem is not a normal mode of oscillation in which all quantities vary exponentially with time. In this sense, it is not a wave. Initially, the field varies exponentially with time with a complex exponent, but the charge densities of various velocity classes of electrons do not. Further, it appears that ultimately even the exponential variation of the field must break down.

It might be objected that the failure of linearity implied by (32) is a product of the nature of the analysis rather than of the actual physical nature of the problem. The formulation used above is a multi-stream formulation, in which we group together electrons which have the same velocity prior to the introduction of the disturbance. It is well known that, in such an analysis, difficulties arise when the ac velocities cause electrons to overtake one another.

We can avoid this difficulty by working with the charge density in phase space (the distribution function) rather than with the charge density in configuration space.

Liouville's theorem[9] states that the total time derivative of the distribution function in the vicinity of a given particle is zero. By expressing the total derivative as a sum of partials, and by expressing the time derivative of the absolute particle velocity w by means of the equation for the particle acceleration, we find that the distribution function $f(t, z, w)$ satisfies the equation

$$\frac{\partial f}{\partial t} + w\frac{\partial f}{\partial z} = \frac{e}{m}(E + F)\frac{\partial f}{\partial w}. \tag{37}$$

In dealing with this equation, w is to be regarded as one of the coordinates, just like t and z. There is no variation of w with t or z; instead, f varies with t, z, and w.

It is assumed that at $t = 0$ we have $f \equiv P(w)$, a function of w alone, where P satisfies (1) and (2). We linearize (37) by replacing $\partial f/\partial w$ by dP/dw. Let $g(t, w)$ be defined by

$$f = P(w) + e^{-j\beta z}g(t, w). \tag{38}$$

With the substitution of (38), where F is as given by (17), the linearized equation becomes

$$\frac{\partial g}{\partial t} - j\beta w g = \left[\frac{e}{m}E^* - v_p\delta(t)\right]\frac{dP}{dw}, \tag{39}$$

where $E = E^*e^{-j\beta z}$. Taking Laplace transforms,

$$(S - j\beta w)\bar{g} = \left[\frac{e}{m}\overline{E^*} - v_p\right]\frac{dP}{dw}. \tag{40}$$

Also we have the equations,

$$\frac{\partial \rho}{\partial t} = -\frac{\partial i}{\partial z}; \qquad i + \epsilon\frac{\partial E}{\partial t} = 0. \tag{41}$$

Letting $\rho = \rho^*e^{-j\beta z}$, $i = i^*e^{-j\beta z}$ and taking Laplace transforms,

$$S\overline{\rho^*} = j\beta\overline{i^*}; \qquad \overline{i^*} + S\epsilon\overline{E^*} = 0. \tag{42}$$

Eliminating $\overline{E^*}$ and $\overline{i^*}$ from (40) and (42),

$$\bar{g} = \frac{\left[\dfrac{e}{m}\dfrac{j\overline{\rho^*}}{\beta\epsilon} - v_p\right]}{(S - j\beta w)}\frac{dP}{dw}. \tag{43}$$

Inverting this equation,

$$g = \left(\frac{e}{m}\frac{j}{\beta\epsilon}\int_0^t \rho^*(\tau)e^{j\beta w(t-\tau)}\,d\tau - v_p e^{j\beta w t}\right)\frac{dP}{dw}. \tag{44}$$

But, from (38) and the definition of the distribution function,

$$\rho^* = -\rho_0\int_{-\infty}^{\infty} g\,dw. \tag{45}$$

Substituting for g from (44) into (45) and performing the integration with respect to w, we have for the bell-shaped distribution,

$$\rho^* = -\left(\omega_p^2\int_0^t \rho^*(\tau)(t - \tau)e^{-\beta u_0(t-\tau)}\,d\tau + j\beta v_p t\rho_0 e^{-\beta u_0 t}\right). \tag{46}$$

This is an integral equation for ρ^* which we may solve by taking Laplace transforms,

$$\overline{\rho^*} = -\frac{(\omega_p^2\overline{\rho^*} + j\beta v_p\rho_0)}{(S + \beta u_0)^2}; \quad \overline{\rho^*} = -\frac{j\beta v_p\rho_0}{[(S + \beta u_0)^2 + \omega_p^2]}. \tag{47}$$

From (42) and (47), we derive the same expression for $\overline{E^*}$ as was found in (24) by the multistream approach. Let us now determine the distribution function perturbation. From (43) and (47),

$$\bar{g} = -\frac{v_p -}{(S - j\beta w)}\frac{dP}{dw} + \frac{\omega_p^2 v_p\dfrac{dP}{dw}}{(S - j\beta w)[(S + \beta u_0)^2 + \omega_p^2]} = \overline{g_1} + \overline{g_2}. \tag{48}$$

The first component g_1 is that caused by the impulsive force F at $t = 0$. We obtain it from (40) by disregarding the first term in the brackets on the right and inverting for $\bar{g}$. This gives

$$g_1 = -v_p e^{j\beta w t}\frac{dP}{dw}. \tag{49}$$

From (48)

$$\overline{g_2} = \frac{\omega_p^2 v_p\dfrac{dP}{dw}}{(S - j\beta w)[(S + \beta u_0)^2 + \omega_p^2]} \tag{50}$$

$$= \frac{\omega_p^2 v_p\dfrac{dP}{dw}}{[\beta^2(u_0 + jw)^2 + \omega_p^2]}\left(\frac{1}{(S - j\beta w)} - \frac{(S + j\beta w + 2\beta u_0)}{[(S + \beta u_0)^2 + \omega_p^2]}\right).$$

Inverting,

$$g_2 = \frac{v_p\omega_p\dfrac{dP}{dw}}{[\omega^2 - \beta^2(w - ju_0)^2]}(\omega_p e^{j\beta w t} - e^{-\beta u_0 t}[\omega_p\cos(\omega_p t) + \beta(u_0 + jw)\sin(\omega_p t)]). \tag{51}$$

Now in linearizing the distribution equation, we replaced $\partial f/\partial w$ by dP/dw in (39). Since f is given by (38), the linearization is valid only if

$$\left|\frac{\partial g}{\partial w}\right| \ll \left|\frac{dP}{dw}\right|, \qquad t > 0 \quad \text{and all} \quad w. \tag{52}$$

From (49) and (51) it follows that

$$\frac{v_p\beta^2\,|\,w - ju_0\,|^2}{|\,\omega_p^2 - \beta^2(w - ju_0)^2\,|}\left|\frac{d^2P}{dw^2} + j\beta t\frac{dP}{dw}\right| \ll \left|\frac{dP}{dw}\right|. \tag{53}$$

---

[9] A derivation of Liouville's theorem direct from Newton's laws, without the use of Hamilton's equations, is to be found in J. R. Pierce, "Theory and Design of Electron Beams," D. Van Nostrand Co., sec. 4.7; 1954.

This implies that both

$$\frac{v_p\beta^2 \,|\, w - ju_0 \,|^2}{|\, \omega_p^2 - \beta^2(w - ju_0)^2 \,|} \left| \frac{d^2P}{dw^2} \right| \ll \left| \frac{dP}{dw} \right|, \quad \text{all} \quad w, \tag{54}$$

a restriction on the initial distribution function, and

$$\frac{\beta v_p t \cdot |\, \beta(w + ju_0) \,|^2}{|\, \omega_p^2 - \beta^2(w - ju_0)^2 \,|} \ll 1. \tag{55}$$

The latter condition is identical with that obtained for the validity of the linearization in the multi-stream approach. Thus, the limitation imposed by nonlinearity is seen to be inherent in the problem, and it does not represent a weakness of the multi-stream method of solution.

We finally discuss an alternate approach to the transform solution. In the distribution function approach, (43) was first inverted to give g; this value of g was then substituted into (45) and the integration with respect to w carried out. This led to an integral equation for the spatial density which was then solved by a further application of Laplace transforms. Alternatively, we could take the transform of (45) to find

$$\begin{aligned} \overline{\rho^*} &= -\rho_0 \int_{-\infty}^{\infty} \bar{g}\, dw \\ &= -\rho_0 \left[\frac{e}{m} \frac{j\overline{\rho^*}}{\beta\epsilon} - v_p \right] \int_{-\infty}^{\infty} \frac{1}{(S - j\beta w)} \frac{dP}{dw}\, dw, \end{aligned} \tag{56}$$

from (43). For the bell-shaped distribution, it is found by contour integration that

$$\int_{-\infty}^{\infty} \frac{1}{(S - j\beta w)} \frac{dP}{dw}\, dw = \begin{cases} \dfrac{-j\beta}{(S + \beta u_0)^2} & R(S) > 0 \\[2ex] \dfrac{-j\beta}{(S - \beta u_0)^2}, & R(S) < 0, \end{cases} \tag{57}$$

where R denotes the real part. Thus it appears that the change of order of integration, with respect to w and with respect to t, in obtaining $\overline{\rho^*}$ from (45) is not valid for all S. From the nature of the Laplace transform and the inversion theorem,[8] the correct transform of ρ^* is given with $R(S) > 0$ and from (56) and (57) we then arrive at (47). A similar procedure may be adopted in the multi-stream approach. Thus (20) may be first integrated with respect to u from $-\infty$ to $+\infty$, and (23) is obtained for $R(S) > 0$.